Analysis of Time-Delay Systems by Series Approximation

A new method of analyzing the behavior of dynamic systems containing time delays is presented. The method, which is based on the construction of a subsidiary system where the delayed state variables are approximated by Taylor series expansion, is simple and easy to apply. The stability and the region of stability of the subsidiary equation correspond closely to that of the original system.

RAJAKKANNU MUTHARASAN

and

REIN LUUS

Department of Chemical Engineering
University of Toronto
Toronto, Canada

SCOPE

Dynamics of several physical systems such as process systems containing recycle streams and process control systems with time-delayed feedback signal are often modeled as differential-difference equations (or differential equations containing delayed states). For proper operation of such systems, knowledge of stability and the region of stability is required.

The main objective of this study is to develop a method to provide information on the stability and the region of stability of dynamic systems described by differential-difference equations. Although successful applications of Liapunov's second method to linear time-delay systems have appeared in the literature (Krasovskii, 1963), relatively few authors have worked with nonlinear problems. The region of stability predicted by the methods based on Liapunov's method such as reported by Seborg and Johnson (1970, 1971) and Landis and Perlmutter (1972) is rather conservative and may be impractical from the application point of view.

In this paper the time-delay terms are approximated by Taylor series expansion, thus reducing the original system equations to a set of ordinary differential equations. It is found that the behavior of this subsidiary system represents very well the behavior of the original system. The procedure is tested using three systems.

CONCLUSIONS AND SIGNIFICANCE

This paper shows that replacing the delayed state variables by Taylor series expansion retains sufficient dynamic features of the original system to enable one to work with the subsidiary system equations to obtain stability information. Examples presented here show that only two or three terms in the series may be adequate to provide reasonable accuracy for practical purposes.

Stability of time-delay systems therefore can be predicted by analyzing the associated subsidiary equations which consist of ordinary differential equations. The proposed method for analyzing the behavior of time-delay systems is thus simple and straightforward for two-dimensional systems. For higher-order systems, the analysis of the system behavior is dependent on one's ability to analyze the behavior of the subsidiary equations which still simplifies considerably the problem.

The presence of transportation lags and recycle streams in process systems has presented dynamic system models of the following form:

$$\begin{aligned}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, x_1(t-\theta_{11}),\ x_1(t-\theta_{12}), \ldots, x_1(t-\theta_{1n}),\\ x_2(t-\theta_{21}),\ x_2(t-\theta_{22}), \ldots, x_2(t-\theta_{2n}), \ldots,\\ x_n(t-\theta_{n1}),\ x_n(t-\theta_{n2}), \ldots, x_n(t-\theta_{nn})) \quad (1)\end{aligned}$$

where $\mathbf{x}$ and $\mathbf{f}$ are n-dimensional vectors and $x_i(t-\theta_{ij})$ are the delayed-state variables. Determination of the behavior of systems of the above class presents a challenging problem of great practical use.

Several authors have extended Liapunov's second method to analyze the stability of linear time-delay systems (Krasovskii 1963; Hale, 1965). Stability analysis for linear time-delay systems with simple nonlinearities has also appeared in the literature (Kitamura et al., 1966).

Recently Seborg and Johnson (1970) presented a method, which was based on Liapunov's argument, for constructing the region of stability of nonlinear systems containing time-delays. The inherent disadvantage of their procedure is that the estimate of the maximum of the first derivative of the state vector leads to a rather conservative estimate of the actual region of stability. Landis and Perlmutter (1972) pointed out the use of Razumikhin theorems (Razumikhin, 1960) to obtain estimates of the maximum of $||\dot{\mathbf{x}}||$. Later they (1973) reported an uncoupling technique which decomposes complex nonlinear time-delay multivariable systems to manageable problems. Their technique enables one to reduce the order of the nonlinear difference-differential equations when some of the constituent equations are linear. Their analysis involves approximating some of the state variables by linear combinations of other delayed states.

There has been a move to reduce linear time-delay systems to ordinary ones through transformation methods for studying optimal control policies (Slatter and Wells, 1972). Development of such techniques for nonlinear time-delay systems would be greatly beneficial since abundant literature is available in the area of stability analysis of nonlinear lumped-parameter systems.

The main purpose of this investigation is to develop and evaluate an approximation method for reducing the nonlinear time-delay system to an ordinary one called the *subsidiary system*. Of particular interest is the accuracy of the stability information obtained from the subsidiary system in relation to the original system.

THEORETICAL DEVELOPMENTS

Consider the continuous, nonlinear time-invariant process model

$$\dot{x}_i = f_i(\mathbf{x}) + \sum_{j=1}^{n} a_{ij}\, x_j(t - \theta_{ij}) \qquad (2)$$

$$i = 1, 2, \ldots, n$$

Let us expand the delayed-state variable $x_j(t-\theta_{ij})$ in terms of a Taylor series

$$x_j(t-\theta_{ij}) = x_j(t) - \theta_{ij}\,\dot{x}_j(t) + \frac{\theta_{ij}^2}{2}\ddot{x}_j(t) \qquad (3)$$

Let us consider only the first two terms in the above expansion and substitute into Equation (2). Upon rearrangement, Equation (2) becomes

$$\dot{\mathbf{x}} = (\mathbf{I}+\mathbf{B})^{-1}\,[\mathbf{f}(\mathbf{x}) + \mathbf{A}\,\mathbf{x}] \qquad (4)$$

where the (i, j)th element of $\mathbf{B}$ *is* $a_{ij}\,\theta_{ij}$. We shall call Equation (4) the associated second-order subsidiary system equation of the original system [Equation (2)]. The term *order* refers to the number of terms in the Taylor series considered in the approximation. The method presented here is not restricted to only two terms. When three terms in the approximation are desired, an estimate of $\dot{x}_j(t-\theta_{ij})$ is required. Expansion of $\dot{x}_j(t-\theta_{ij})$ in a Taylor series gives (retaining only two terms)

$$\dot{x}_j(t-\theta_{ij}) = \dot{x}_j(t) - \theta_{ij}\,\ddot{x}_j(t) \qquad (5)$$

An expression for $\ddot{x}_j(t)$ is easily obtained by differentiating Equation (2) once; thus

$$\ddot{x}_j = \sum_{l=1}^{n} f_{jxl}(\mathbf{x})\,\dot{x}_l + \sum_{l=1}^{n} a_{jl}\,\dot{x}_l(t-\theta_{jl}) \qquad (6)$$

The estimate for $\dot{x}_j(t-\theta_{ij})$ from Equation (5) may now be substituted into Equation (6) to obtain

$$\ddot{\mathbf{x}} = (\mathbf{I}+\mathbf{B})^{-1}\,(\mathbf{f}_{\mathbf{x}} + \mathbf{A})\,\dot{\mathbf{x}} \qquad (7)$$

where the (i, j)th element of $\mathbf{f}_{\mathbf{x}}$ is given by $\partial f_i(\mathbf{x})/\partial x_j$. The right-hand side of Equation (7) can be substituted in the series expansion of $x_j(t-\theta_{ij})$ and, as before, substitution of the resulting expression into the system equation [Equation (2)] yields

$$\dot{\mathbf{x}} = (\mathbf{I}+\mathbf{G})^{-1}\,(\mathbf{f}(\mathbf{x}) + \mathbf{A}\,\mathbf{x}) \qquad (8)$$

where $\mathbf{G}$ is given by

$$\mathbf{G} = \mathbf{B} - \mathbf{C}(\mathbf{I}+\mathbf{B})^{-1}\,[\mathbf{f}_{\mathbf{x}} + \mathbf{A}] \qquad (9)$$

and the (i, j)th element of $\mathbf{C}$ is $\frac{a_{ij}\,\theta_{ij}^2}{2}$. Equation (8) constitutes the associated third-order subsidiary system equation of the original system. Thus, there is no difficulty in constructing higher order subsidiary system equations.

Since the amount of algebraic manipulations increases significantly as the order of the subsidiary system is increased, it is of practical interest to find how well the original system is represented by the lowest order approximation. Therefore we consider three examples to illustrate and test the proposed procedure.

Example 1: A Linear System

To illustrate the proposed procedure, we first consider a linear system for which exact analysis is possible. Consider the system described by

$$\dot{x}_1 = x_2 \qquad (10)$$

$$\dot{x}_2 = -bx_1 - (1+b)x_2 - Kbx_1(t-a) \qquad (11)$$

where a and b are positive constants and the delay term is given by $x_1(t-a)$. For simplicity, consider only two terms in the Taylor series expansion of $x_1(t-a)$,

$$x_1(t-a) \simeq x_1(t) - a\dot{x}_1(t) \qquad (12)$$

which, upon substitution of Equation (10), becomes

$$x_1(t-a) \simeq x_1(t) - ax_2 \qquad (13)$$

Substitution of Equation (13) into Equation (11) gives the subsidiary system equations

$$\dot{x}_1 = x_2 \qquad (14)$$

$$\dot{x}_2 = -b(1+K)x_1 + (abK - b - 1)x_2 \qquad (15)$$

For stability of the subsidiary system, the following conditions must hold:

$$b + Kb > 0 \qquad (16)$$

$$abK - b - 1 < 0 \qquad (17)$$

It is readily seen that the ultimate value of K beyond which the system becomes unstable is given by

$$K = \frac{1+b}{ab} \qquad (18)$$

By including three terms in the Taylor series for $x_1(t-a)$, we obtain the third-order subsidiary system equations

$$\dot{x}_1 = x_2 \qquad (19)$$

$$\dot{x}_2 = (a^2bKp - 2Kp - b)x_1 + (2Kap + Ka^2pb + Ka^2p - b - 1)x_2 \qquad (20)$$

where

$$p = \frac{b}{2 + Ka^2b} \qquad (21)$$

For stability it is necessary that

$$Kpa^2b - b - 2Kp < 0 \qquad (22)$$

$$1 + b - 2Kap - Ka^2p(1+b) > 0 \qquad (23)$$

A similar analysis using four terms in the Taylor series yields the following conditions for stability (with $b = 1.0$):

$$\frac{(1+K)}{\alpha} > 0 \qquad (24)$$

$$-\frac{\beta}{\alpha} > 0 \qquad (25)$$

where

$$\alpha = 1 + \frac{Ka^2}{2} - \frac{a^3K(aK-2)}{6(1+0.5\,Ka^2)} \qquad (26)$$

$$\beta = aK - 2 - \frac{Ka^3(1+K)}{6(1+0.5\,Ka^2)} \qquad (27)$$

It is instructive to compare the values of ultimate proportional constant predicted by the various approximations to those calculated by exact methods, such as Bode analysis. The results of these computations, for various delay factors a, are presented in Table 1. It is interesting to note that the second order approximation predicts the ultimate K within 10% of the exact value. In this specific example, we observe that the predicted ultimate K by the second- and third-order approximations are the same since the criterion for stability predicted by the third-order approximation, Equations (22) and (23), simplifies to that of the second-order approximation Equation (18).

The fourth-order approximation, however, gives the result to within 0.3%.

Now let us turn our attention to two systems encountered in chemical engineering which have been analyzed by other methods.

Example 2: A Nonlinear System

Landis and Perlmutter (1972) considered the autorefrigerated reactor, the mathematical model of which was originally presented by Luyben (1966), to test their direct method for determining the region of stability of systems described by nonlinear differential-difference equations. The reactor, in which the heat generated by an exothermic first-order irreversible chemical reaction is removed by the evaporation of the solvent, was modeled as

$$\frac{dT}{dt} = 0.833(294 - T) + 444\,r - 2.93 \times 10^{-7}\,hq \qquad (28)$$

$$\frac{dc}{dt} = 0.833(0.5 - c) - r \qquad (29)$$

Where the reaction rate, r, is given by

$$r = 7.08 \times 10^{10}\,c\,\exp(-8393/T) \qquad (30)$$

The temperature dependence of latent heat of vaporization is expressed by

$$h = \begin{cases} 2.11 \times 10^6\sqrt{1 - 0.00273\,T}, & T < 366.5°\text{K} \\ 0 & T \geq 366.5°\text{K} \end{cases} \qquad (31)$$

The vapor stream flow rate is controlled in proportion to the temperature in the reactor. Due to the presence of time-delay in the feedback loop, the flow rate is changed according to the following relationship:

$$q = 331 + K(T(t-a) - 333) \qquad (32)$$

where a is the delay time. Equations (28) through (32) describe the dynamics of the autorefrigerated reactor system.

Expanding the time-delay term and retaining only the first two terms,

$$T(t-a) \simeq T - a\frac{dT}{dt} \qquad (33)$$

The above estimate of the time-delay term can be substituted into the original equations, Equations (28) and (29), and the following associated second-order subsidiary system equations result:

$$\frac{dT}{dt} = \frac{0.833(294 - T) + 444r - 9.67 \times 10^{-5}\,h - 2.93 \times 10^{-7}\,hkT + 9.75 \times 10^{-5}\,hK}{1 - 2.93 \times 10^{-7}\,ahK} \qquad (34)$$

$$\frac{dc}{dt} = 0.833(0.5 - c) - r \qquad (35)$$

It is illustrative to compare the phase-plane diagrams of the subsidiary system [Equations (34) and (35)] and the original system [Equations (28) and (29)]. We note that the solution of the original system equations requires a set of initial curves while only an initial state is required for the subsidiary system. The original system was integrated by choosing for the initial curves the constant initial conditions. The state of the original system at $t = a$ was chosen as the initial condition of the subsidiary system.

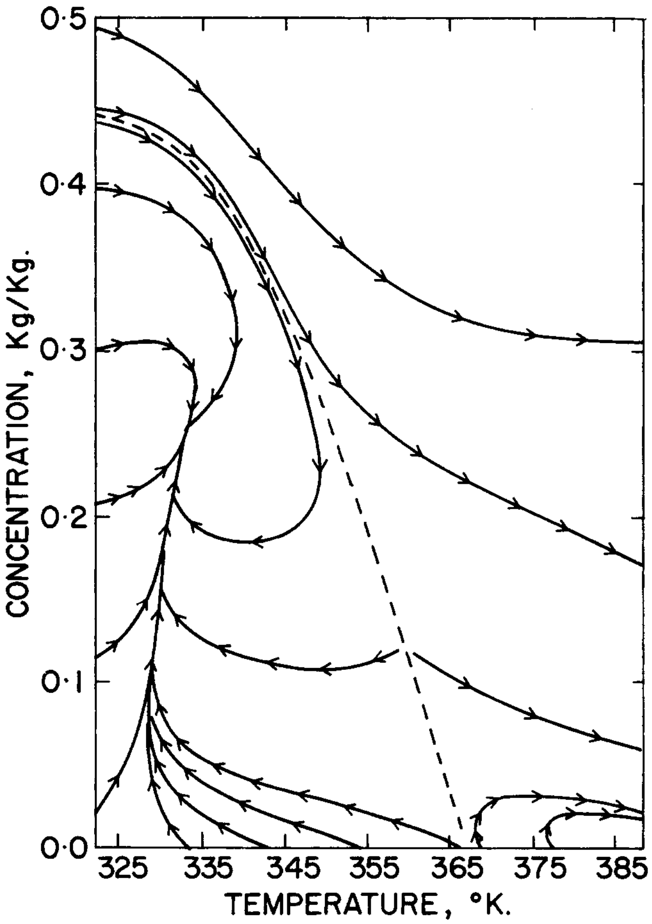

Fig. 1. Phase-plane diagram for the autorefrigerated reactor using second order approximation: a = 0.004 h. K = 81.65 kg/h/°K [= 100 lb/h/°F]; - - - encloses the region of stability.

Table 1. Comparison of Exact and Predicted Ultimate Proportional Constants with $b = 1$

Delay factor a	Exact analysis	Order of approximations: Second order	Third order	Fourth order
0.1	20.7282	20.0	20.0	20.6770
0.2	10.6957	10.0	10.0	10.6859
0.3	7.3574	6.6667	6.6667	7.3599
0.4	5.6911	5.0	5.0	5.6993
0.5	4.6931	4.0	4.0	4.7040
0.6	4.0292	3.3333	3.3333	4.0408
0.7	3.5561	2.8571	2.8571	3.5671

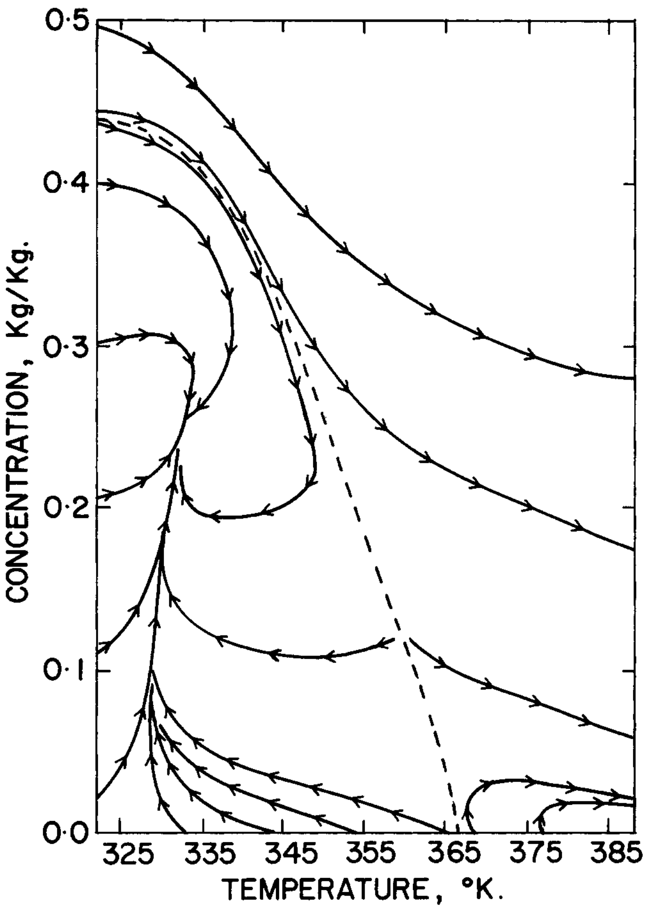

Fig. 2. Phase-plane diagram for the autorefrigerated reactor using original system equations; a = 0.004 h. K = 81.65 kg/h/°K [= 100 lb/h/°F]; - - - encloses region of stability.

This type of restriction on the choice of initial condition guarantees that the initial curves specified are consistent with the original system equations.

In Figures 1 and 2 we present the phase-plane diagrams of the subsidiary and the original system when $a = 0.004$ h and $K = 81.65$ kg/h/°K [which is equivalent to $K =$ 100 lb/hr/°F]. The dashed-line encloses the region of stability. As can be seen from Figure 3, the region of stability of the subsidiary system is very close to the region of stability of the system. Also the trajectories are almost identical. In the same figure the region of stability predicted by Landis and Perlmutter (1972) is also presented.

As the time delay is increased, the trajectories in the stable region start to deviate slightly. The region of stability, however, is still predicted very well as is shown in Figure 4 for the example with $a = 0.05$ hr.

Example 3: CSTR with Proportional Control

By applying the proposed procedure to the reactor problem of Seborg and Johnson (1970, 1971), we obtained a considerably larger region of stability. Furthermore, the phase plane plots of the subsidiary equation and of the actual system are very similar. This gives further support that reliable stability information can be readily obtained for such systems.

DISCUSSION

The three examples presented here show that the use of Taylor series expansion provides a useful means of analyzing the behavior of the original system where delay terms

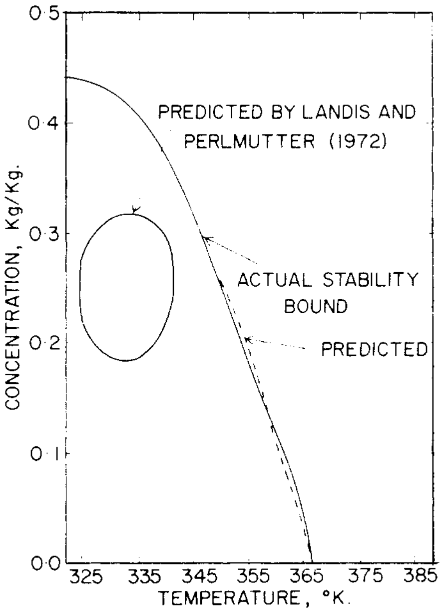

Fig. 3. Comparison of predicted and actual region of stability for the autorefrigerated reactor system; a = 0.004 h, K = 81.65 kg/h/°K [= 100 lb/h/°F].

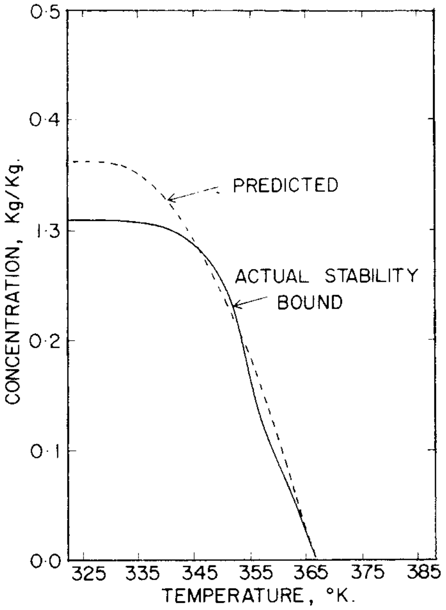

Fig. 4. Comparison of predicted and exact region of stability; a = 0.05 h, K = 81.65 kg/h/°K [= 100 lb/h/°F].

are present. Although the information obtained from the subsidiary system may not be exact, it still provides an invaluable guide for the analysis of time-delay systems, just as the averaging technique is useful as a guide for the analysis of two-dimensional nonlinear systems (Luus and Lapidus, 1972).

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putation facilities of University of Toronto were used for this investigation.

NOTATION

a, = delay time
a_{ij} = constants
$\mathbf{A}$ = coefficient matrix whose (i, j)th element is a_{ij}
$\mathbf{B}$ = constant matrix whose (i, j)th element is $a_{ij}\,\theta_{ij}$
c = reactant concentration
$\mathbf{C}$ = constant matrix whose (i, j)th element is $a_{ij}\,\theta_{ij}^2/2$
$\mathbf{G}$ = matrix defined by Equation (9)
$\mathbf{I}$ = identity matrix
K = proportional controller constant
p = expression given in Equation (21)
q = vapor stream flow rate
r = reaction rate; see Equation (30)
t = time
T = temperature
$x_i(t)$ = state variable
$\mathbf{x}(t)$ = state vector

Greek Letters

α = expression given in Equation (26)
β = expression given in Equation (27)
θ_{ij} = delay time

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Transport of Mass in Ternary Liquid-Liquid Systems

The diffusional behavior of the ternary partially miscible liquid system acetonitrile-benzene-*n*-heptane is investigated at 25°C using the diaphragm cell technique. The mutual diffusivity of the partially miscible binary system acetonitrile-*n*-heptane is found to be a strong function of concentration; the concentration dependence is described using a linear relationship. The diffusion coefficient matrix of the ternary system acetonitrile-benzene-*n*-heptane is determined at a number of points, and the ternary diffusion matrices are found to be significantly coupled with the off-diagonal elements of about 10 to 40% of the main diffusion coefficients.

ANIL SETHY

and

HARRY T. CULLINAN, JR.

Department of Chemical Engineering
State University of New York at Buffalo
Buffalo, New York 14214

SCOPE

Most of the previous experimental work on interfacial mass transfer in ternary heterogeneous liquid systems has been restricted to mutually saturated bulk phases with only one transferring solute. However, practical mass transfer processes usually occur between unsaturated bulk phases and sometimes involve the simultaneous transfer of more than a single constituent. Furthermore, although some information has accumulated on the diffusion coefficients of ternary liquid systems, few parallel direct measurements of the corresponding mass transfer coefficients in such systems have been attempted. It is well known that the diffusive interactions in thermodynamically nonideal systems can be very pronounced; therefore, it is of interest to investigate the corresponding behavior of the mass transfer coefficients. In addition, mass transfer in liquid-liquid systems may be accompanied by interfacial instabilities. These alter the hydrodynamic conditions in the vicinity of the interface and thus affect the transfer rate. This means that conclusions cannot be drawn concerning the